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Logarithmic lifts of the family $\lambda z e^z$

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1 Introduction

We give a 1-parameter family of entire functions, which gives an example where a Baker domain changes to infinite number of wandering domains. This example is interesting from the viewpoint of the Teichmüller theory recently introduced by McMullen and Sullivan [5]. See [4] for some details.

The family we considered is the logarithmic lifts of

$$\mathcal{E} = \{f_\lambda(z) = \lambda z e^z\}$$

with $1/e \leq \lambda \leq e$.

More explicitly, we consider the family of entire functions

$$\mathcal{L} = \{g_\lambda(z) = z + e^z + \log \lambda\}.$$

Actually, $g(z)$ is determined up to $2k\pi$ ($k \in \mathbf{Z}$), and we see that the change of additive constant makes a dramatic change of the dynamics.

All the figures in this note are produced by Professor S. Morosawa.

2 The family $\mathcal{E}$

Every element f in $\mathcal{E}$ has the asymptotic value 0, and the critical value $-\lambda/e$. The fixed points of f are 0 and $-\log\lambda$ (which we regard as a real). The Teichmüller space $\mathrm{Teich}(\mathbf{C}, f)$ of the dynamics by f is at most one-dimensional. (Cf. also [3].)

The case I: $\lambda = 1/e$.

In this case, 0 is an attractive fixed point, and $-\log\lambda = 1$ is a repelling one. Let D be the immediate attractive basin of 0. Then $-\lambda/e = -1/e^2$ is contained in D, and the Teichmüller space Teich$([D], f)$ of the dynamics by f restricted on the grand orbit $[D]$ of D is that of a once-punctured torus, and hence one-dimensional.

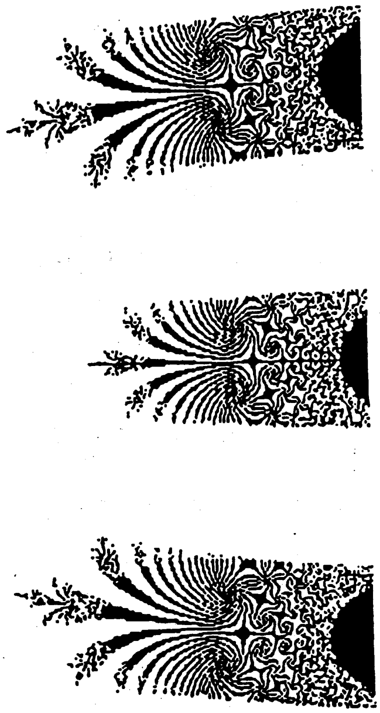

Figure 1: $\lambda = 1/e$

The case II: $1/e < \lambda < 1$.

In this case, the situation is the same as in the case I.

The case III: $\lambda = 1$.

In this case, 0 is a parabolic fixed point, and $-\log\lambda = 0$. Let D be the immediate attractive basin of 0. Then $-\lambda/e = -1/e$ is contained in D, and $\mathrm{Teich}([D], f)$ is that of the thrice-punctured sphere, and hence trivial.

Figure 2: $\lambda = 1$

The case IV: $1 < \lambda < e$.

In this case, 0 is a repelling fixed point, and $-\log\lambda$ is an attracting one. Let D be the immediate attractive basin of $-\log\lambda$. Then $-\lambda/e$ is contained in D, and $\mathrm{Teich}([D], f)$ is again that of a once-punctured torus, and hence one-dimensional.

The case V: $\lambda = e$.

In this case, 0 is a repelling fixed point, and $-\log\lambda = -1$ is an attracting one. Since $-\lambda/e = -1$, -1 is super-attracting. Let D be the immediate attractive basin of -1. Then the dynamics of f on $[D - \{-1\}]$ is not discrete, and Teich($[D], f$) is trivial.

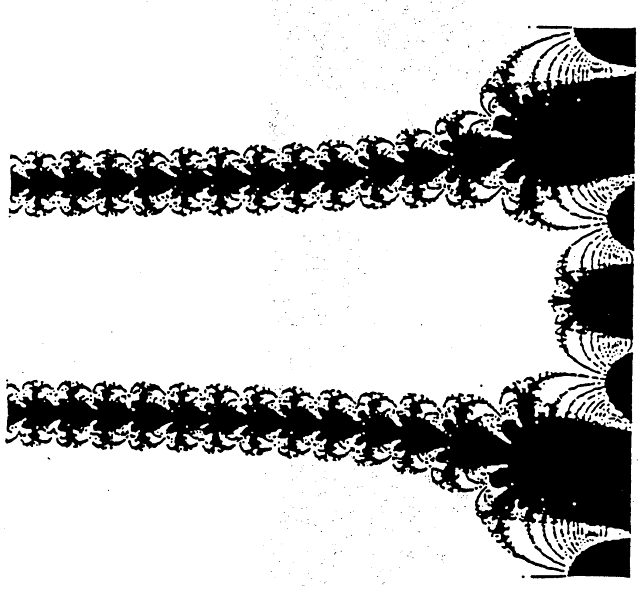

Figure 3: $\lambda = e$

3 The logarithmic lift

A logarithmic lift g of an endomorphism f of $\mathbf{C}^*$ is an entire function satisfying that

$$e^{g(z)} = f(e^z).$$

The case I: $\lambda = 1/e$.

In this case, a logarithmic lift of $f_{1/e}$ is

$$g(z) = z + e^z - 1.$$

Note that, by taking affine conjugates of $f_{1/e}$ and g, this g is equivalent to

$$z + e^{-z} + 1.$$

This is a famous example of Fatou, which has a Baker domain D.

The Teichmüller space Teich($[D], g$) is that of $\mathbf{C} - \mathbf{Z}$, and hence infinite-dimensional. This situation is not changed when we take other logarithmic lifts.

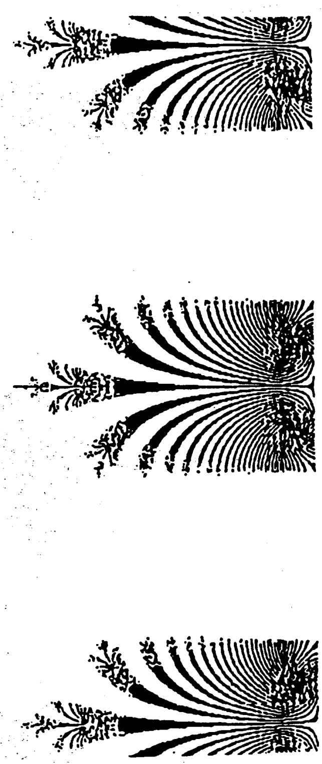

Figure 4: $\log \lambda = -1$

The case II: $e < \lambda < 1$.

In this case, a logarithmic lift of f_λ is

$$g(z) = z + e^z + \log \lambda.$$

And the situation is the same as in the case I.

The case III: $\lambda = 1$.

In this case, a logarithmic lift of f_1 is

$$g(z) = z + e^z.$$

This g has infinitely many Baker domains. Let D be any one of them. Then Teich($[D], f$) is that of the thrice-punctured sphere, and hence trivial.

On the other hand, if we take

$$g(z) = z + e^z + 2\pi i$$

as a logarithmic lift, g has a wandering domain D. And Teich($[D], g$) is that of $\mathbf{C} - \mathbf{Z}$, and hence infinite-dimensional.

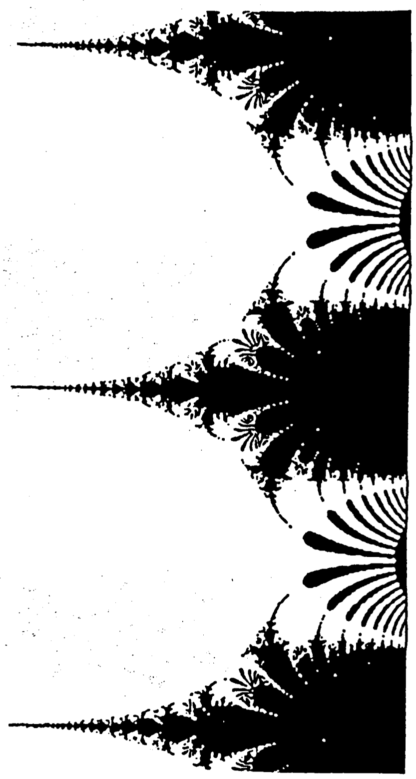

Figure 5: $\log\lambda = 0$

The case IV: $1 < \lambda < e$.

In this case, a logarithmic lift of f_λ is

$$g(z) = z + e^z + \log\lambda.$$

g has no wandering domains and no Baker domains.

On the other hand, if we take

$$g(z) = z + e^z + \log\lambda + 2\pi i$$

as a logarithmic lift, g has a wandering domain D. And Teich($[D], g$) is that of $\mathbf{C} - \mathbf{Z}$, and hence infinite-dimensional.

The case V: $\lambda = e$.

In this case, a logarithmic lift of f_e admitting a wandering domain is

$$g(z) = z + e^z + 1 - 2\pi i.$$

This is equivalent to the example of Baker:

$$g(z) = z - e^z + 1 + 2\pi i.$$

Let D be the wandering domain of g. Then the dynamics of g on $[D - \{-1\}]$ is not discrete, and Teich($[D], g$) is trivial.

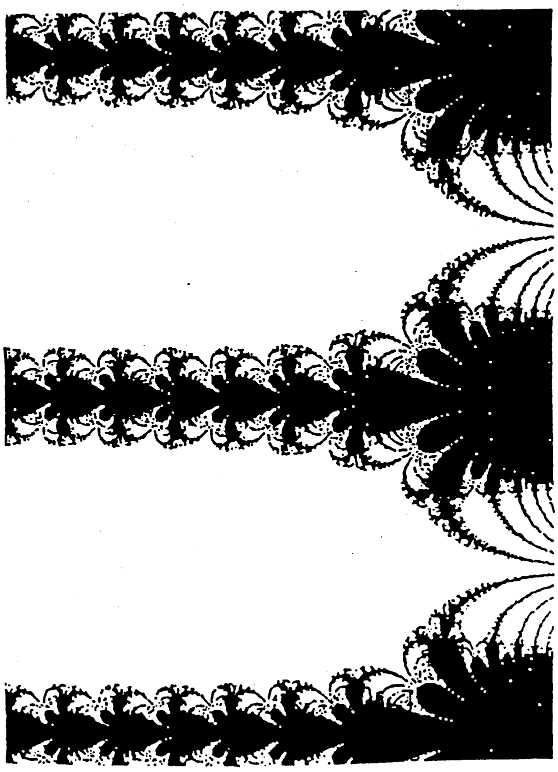

Figure 6: $\log\lambda = 1$

References

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